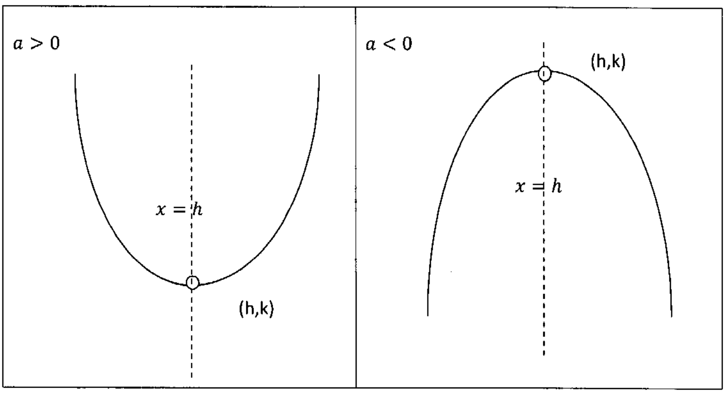
Forms of quadratic functions:

General form

General	Standard (completed square)
$f(x) = ax^2 + bx + c$	$f(x) = a(x-h)^2 + k$ ← Vertex: (h, k) Axis of symmetry: $x = h$ $h = -\frac{b}{2a}$ or just complete the square $k = f(h)$



Example: P. 312 #48

Sketch the graph of $f(x) = -3x^2 + 6x + 2.$ Put in standard form

List and draw the vertex, axis of symmetry, and intercepts

Put in the form $f(x) = a(x-h)^2 + k$

$h = -\frac{b}{2a} = -\frac{6}{2\cdot(-3)} = 1$

$k = f(h) = f(1) = -3\cdot 1^2 + 6\cdot 1 + 2 = -3+6+2 = 5$

Standard form: $f(x) = -3(x-1)^2 + 5$

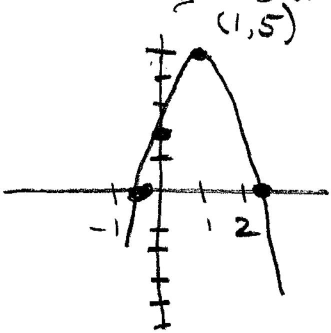
Vertex: $(1,5) = (h,k)$

Axis of symmetry: $x = 1$ $x = h$

Y-intercept: $x=0$ $f(0) = 2$

X-intercepts: solve $y = f(x) = 0$

$-3(x-1)^2 + 5 = 0$



$-3(x-1)^2 = -5$

$(x-1)^2 = \frac{+5}{+3}$

$x - 1 = \pm\sqrt{\frac{5}{3}}$

$x = 1 \pm \sqrt{\frac{5}{3}}$

$1+\sqrt{\frac{5}{3}}, \quad 1-\sqrt{\frac{5}{3}}$

$2+?$ $0-?$

Example: P. 312 #48

Sketch the graph of $f(x) = -3x^2 + 6x + 2$.

List and draw the vertex, axis of symmetry, and intercepts

Put in the form $f(x) = a(x-h)^2 + k$

$$h = -\frac{b}{2a} = -\frac{6}{2(-3)} = -\frac{6}{-6} = 1$$

$$k = f(h) = f(1) = -3(1)^2 + 6(1) + 2 = -3 + 6 + 2 = 5$$

Standard form: $f(x) = -3(x-1)^2 + 5$

Vertex: (1,5)

Axis of symmetry: $x = 1$

Y-intercept: $f(0) = 2$

X-intercepts: solve $f(x) = 0$

$$f(x) = -3(x-1)^2 + 5 = 0$$

$$-3(x-1)^2 = -5$$

$$(x-1)^2 = \frac{5}{3}$$

$$x - 1 = \pm\sqrt{\frac{5}{3}}$$

$$x = 1 \pm \sqrt{\frac{5}{3}}$$

POLYNOMIALS

- Domain
- Factors, zeroes, multiplicity
- Power functions and end behavior
- Graphing polynomials

Degree	Example	Form	Graph
0	$f(x) = 3$	$f(x) = a_0$	Horizontal line
1	$f(x) = 2x - 1$	$f(x) = a_1 x + a_0$	Line
2	$f(x) = -3x^2 + 6x + 2$	$f(x) = a_2 x^2 + a_1 x + a_0$ Or $f(x) = ax^2 + bx + c$	Parabola
3	$f(x) = x^3 + 2x^2 - 1$	$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$	
n 35	$f(x) = x^{35} - 2x^{34} \ldots + 11$	$f(x) = a_n x^n + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$	

Only *non-negative integer* powers of x can appear

The *degree n* is the highest power that appears (with non-zero *coefficient*.) The coefficients are $a_0, a_1, \ldots, a_n$

P. 330 12, 14, 18, 20, 22

	Polynomial?	Degree?
12. $f(x) = 5x^2 + 4x^4$	YES	4
14. $h(x) = 3 - \frac{1}{2}x^1$	YES	1
16. $f(x) = x(x - 1)$	$x^2 - x$ YES	2
18. $h(x) = \sqrt{x}(\sqrt{x} - 1)$	$(\sqrt{x})^2 - \sqrt{x}$ $x - \sqrt{x}$ $x - x^{1/2}$	NO
20. $F(x) = \dfrac{x^2 - 5}{x^3}$	$x^{-1} - 5x^{-3}$	NO
22. $G(x) = -3x^2(x + 2)^3$	YES	~~YES~~ 5

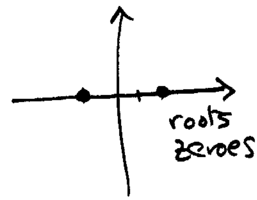
What is the domain of ***any*** polynomial function? **All real numbers!** $(-\infty, \infty)$.

$12 = 2 \cdot 2 \cdot 3$

POLYNOMIALS: Factors, zeros (roots) and MULTIPLICITY

Example: $g(x) = x^2 - x - 2$ is a polynomial of degree 2

Factor: $g(x) = (x-2)(x+1)$ Factored form:



The *factors* of $g(x)$ are $(x-2)$ and $(x+1)$

The *zeros* or *roots* of $g(x)$ are 2 and -1, because $g(2) = 0$ and $g(-1) = 0$.

If $g(x)$ is a polynomial, then these are *equivalent* statements:

$g(r) = 0$	r is a zero of $g(x)$ root	r is an x-intercept of the graph of $g(x)$	$(x - r)$ is a factor of $g(x)$

Example: $f(x) = x^3 + 2x^2 + x$ degree: 3

Factor: $f(x) = x(x^2 + 2x + 1) = x(x+1)^2 = x(x+1)(x+1)$

$(x-0)(x-(-1))(x-(-1))$

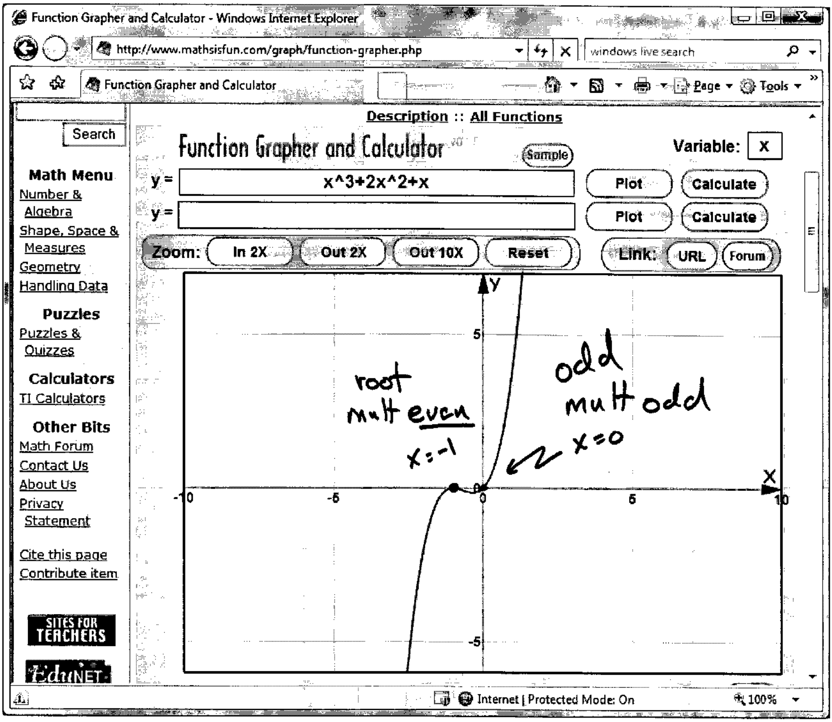
Zeros of f(x) are: -1 and 0 (why?)

Factors of $f(x)$ are x and $(x+1)$

The *multiplicity* of the zero -1 is *2*.

The multiplicity of a root r is the *exponent* that appears with the factor $(x - r)$.

Now graph $f(x)$



Multiplicity of r is EVEN	Multiplicity of r is ODD
Graph of $f(x)$ touches but does not cross the x-axis at r	Graph of $f(x)$ crosses the x-axis at r
$f(x)$ does *not* change sign at r	$f(x)$ changes sign at r

$f(x) = x^3 + 2x^2 + x$ has degree 3

Factored: $f(x) = x(x+1)^2 = (x)(x+1)(x+1)$

This polynomial is *completely factored:* degree 3, and there are 3 factors, each of degree 1.

Not all polynomials can be completely factored – using real numbers.

Example: $f(x) = (x-2)(x^2+1)$ is of degree 3 but has only one real root: $x = 2$.

Does it have any *non-real* roots? Factor $x^2 + 1 = (x - i)(x + i)$

$x^2 + 1 = 0$

$x^2 = -1$

$x = \pm\sqrt{-1} = \pm i$

It turns out that all polynomials CAN be factored using complex numbers.

(but we're not going to go further on this topic).

P. 331 #40 Construct a polynomial of degree 3 with zeroes -4, 0, 2

r is a zero
$(x-r)$ factor

$f(x) = (x+4)(x-2)x$ degree 3.

P. 331 #44 Construct a polynomial of degree 3 with zeroes -2 (multiplicity 2) and 4

$f(x) = (x+2)^2(x-4)$

Factor $f(x) = -x^2(x^2-4)(x-5)$ as far as possible. What are the zeroes of $f(x)$?

degree 5

$-x^2(x^2-4)(x-5)$

$-x^4$

$-x^5$

Factoring: $x^2 = x \cdot x$
$= (x-0)(x-0)$

$(x^2-4) = (x-2)(x+2)$
$= (x-2)(x-(-2))$

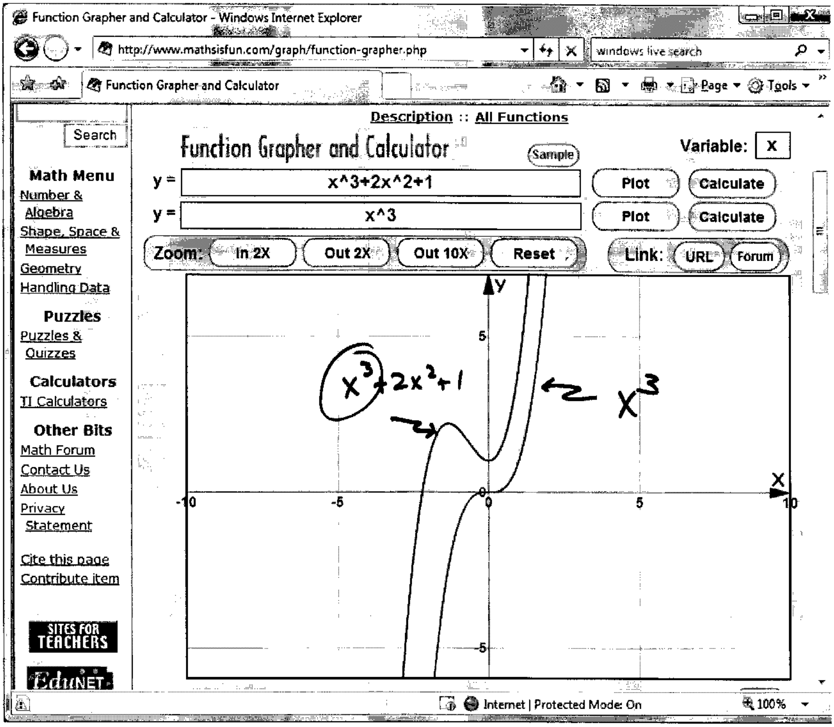
$(x-5)$

$f(x) = -(x-0)(x-0)(x-2)(x-(-2))(x-5)$

Zeroes: 0 (mult 2), +2, −2, +5

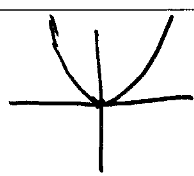
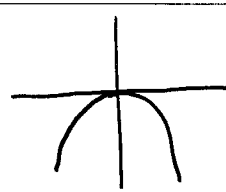
END-BEHAVIOR $x \to \infty$ $x \to -\infty$

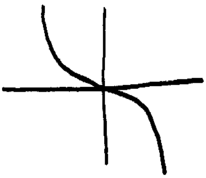
Example: graph $f(x) = x^3 + 2x^2 + 1$ and $g(x) = x^3$



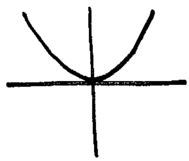
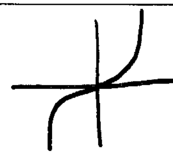
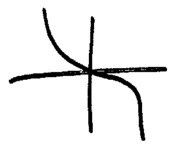
Behavior for large positive and large negative values of x ("end behavior") is determined by the behavior of the term with the largest exponent

What is the end behavior of

$f(x) = x^4$ x^{even} 	$f(x) = -x^2$ $-x^{even}$ 

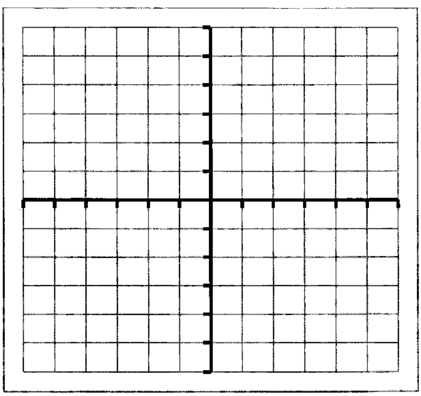
$f(x) = -x^3$ $-x^{odd}$ 	$f(x) = 2x^5$

End behavior of $f(x) = ax^n$ (this is called a *power function*)

	n is even	N is odd
a>0	$x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to \infty$ 	$x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to -\infty$ 
a<0	$x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to -\infty$ 	$x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to \infty$ 

TRANSFORMATIONS OF POWER FUNCTIONS

P. 331 #32 $f(x) = (x+2)^4 - 3$



Graphing polynomials

P. 331 #48 $f(x) = 2(x-3)(x+4)^3$

a. Factor as far as possible and find x-intercepts (zeroes).

b. Find how the graph behaves for each zero

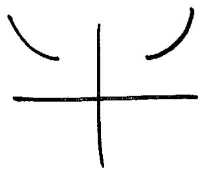
Zero	Multiplicity	Behavior
3	1 odd	crosses y-axis
−4	3	crosses x-axis

c. Find the degree and determine the end behavior degree: 4

Approach: we only care about the term of the highest power

$f(x) = 2(x-3)(x+4)^3 = 2(x-3)(x^3+\cdots) = 2x^4+\cdots$

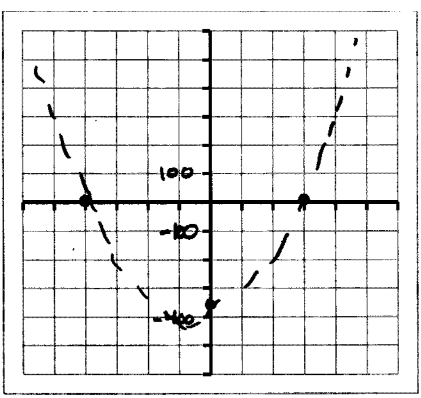
So $f(x)$ behaves like $2x^4$.



y intercept: $x=0$

$f(0) = 2\cdot(-3)(4^3) = -6\cdot 64$

$= -384$



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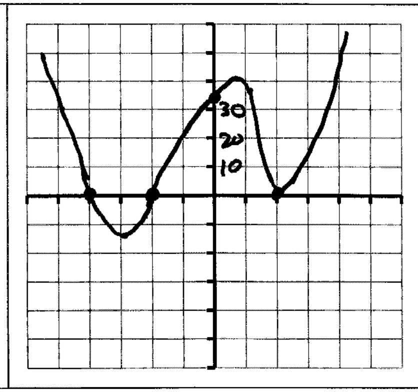
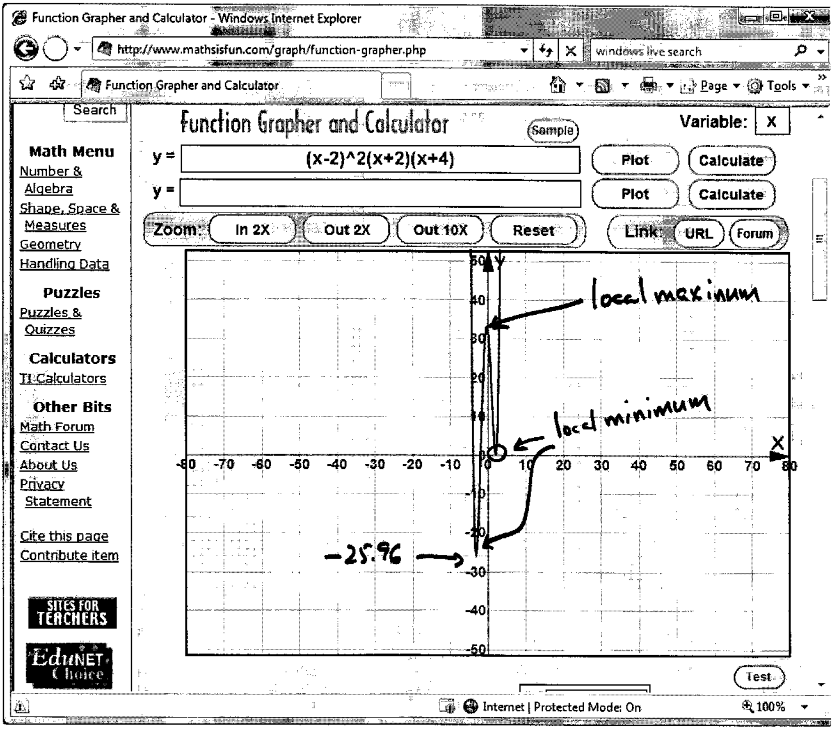
P. 331 #76 $f(x) = (x-2)^2(x+2)(x+4)$ already factored?

Degree	
x-intercepts and behavior	2 grazes/bounces ← −2 through −4 through
y-intercept	$f(0) = (-2)^2(2)(4) = 32$
Graph f with a utility	See next page
Find local maxima and minima	Minima: x=-3.19, f(x)=-25.96 x=2, y=0 Maxima: x=-0.31, y=33.27 skip End: degree 4 $+x^4$
Graph by hand	
Range	
Increasing	
Decreasing	



range $(-25.96, +\infty)$

RATIONAL FUNCTION

Is the ratio of 2 polynomials: $R(x) = \frac{p(x)}{q(x)}$

- Domain
- Revisiting $f(x) = \frac{1}{x}$
- End-behavior
- Asymptotes
- Graphing

DOMAIN

The domain of a polynomial function is $(-\infty, \infty)$.

What is the domain of a rational function $R(x) = \frac{p(x)}{q(x)}$?

What can "go wrong?"

The domain of $R(x) = \frac{p(x)}{q(x)}$ is $\{x: q(x) \neq 0\}$ - that is, $denominator \neq 0$

P. 344 #14: What is the domain of $G(x) = \frac{6}{(x+3)(4-x)}$

Is this a rational fn? YES

$p(x) = 6$ degree 0 $q(x) = (x+3)(x-4)$ degree 2

$0 = q(x) = (x+3)(x-4)$ $x = -3, x = 4$

Domain: $\{x: x \neq -3, x \neq 4\}$

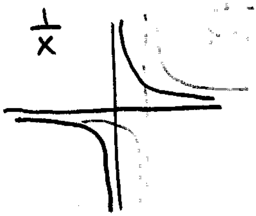
What is the domain of $F(x) = \frac{x+1}{x^2-1}$? $\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)}$

$q(x) = (x+1)(x-1)$

Domain: $\{x: x \neq -1, x \neq 1\}$

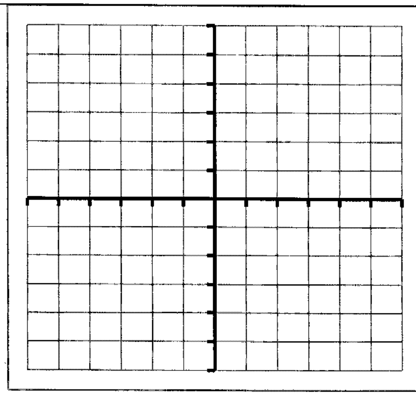
In the domain: $F(x) = \frac{\cancel{x+1}}{\cancel{(x+1)}(x-1)} = \frac{1}{x-1}$

$\frac{1}{x}$

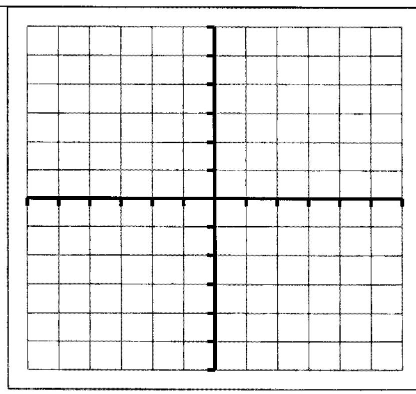


$\frac{1}{x}$

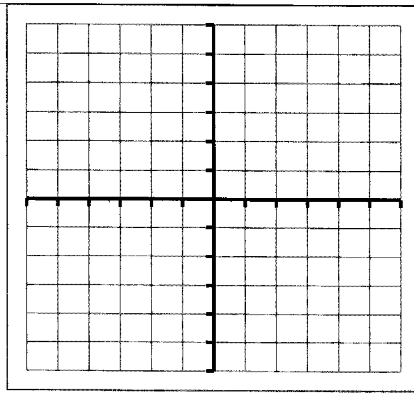
Revisiting $f(x) = \frac{1}{x}$

Domain?	
Range?	
Intercepts?	
Even / odd?	

Graph $f(x) = \frac{1}{x^2}$

Domain?	
Range?	
Intercepts?	
Even / odd?	

Graph $f(x) = \frac{1}{x^3}$

Domain?	
Range?	
Intercepts?	
Even / odd?	

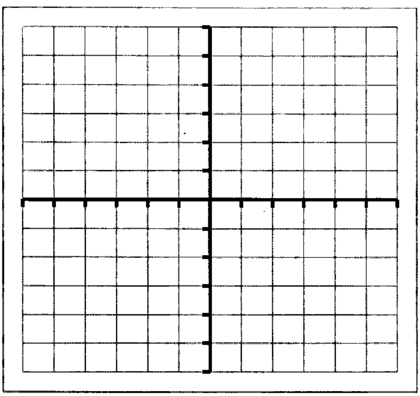
P. 345 #34

Graph $g(x) = \frac{2}{(x+2)^2}$

Transformation

Starting function?

What transformations?



ASYMPTOTES

Vertical asymptotes

$R(x) = \frac{p(x)}{q(x)}$ If $q(c) = 0$ and $p(c) \neq 0$ then the graph has a vertical asymptote $x = c$ (see example above).

ALSO, if $q(c) = 0$ and $p(c) = 0$, the graph MAY have a vertical asymptote. Remove the common factor(s) from both numerator and denominator, and evaluate the result for $x = c$.

Examples

$F(x) = \frac{x+1}{x^2-1}$. Then $p(-1) = q(-1) = 0$

Find common factors:

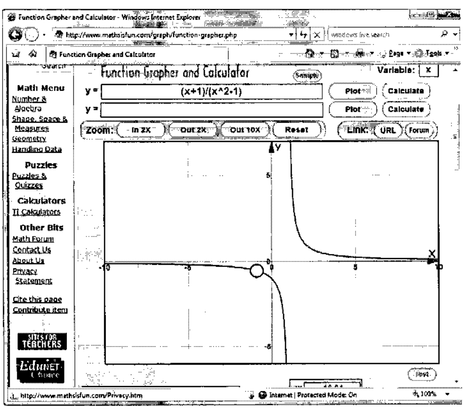
$\frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}$ for $x \neq -1$

Evaluate this at $x = -1$: $\frac{1}{-1-1} = -\frac{1}{2}$

Outcome: $F(x)$ behaves like $\frac{1}{x-1}$ except at $x = -1$, where it is not defined (has a "hole")

Domain of $F(x)$?

Range?



Example: $G(x) = \frac{(x+2)}{x^2+4x+2}$

$p(-2) = q(-2) = 0$

$$\frac{(x+2)}{x^2+4x+2} = \frac{(x+2)}{(x+2)(x+2)} = \frac{1}{x+2}$$

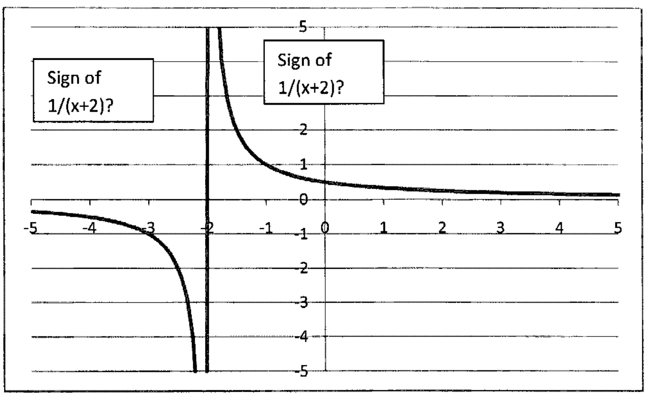
This is not defined for $x = -2$. But as $x \to -2$, the denominator approaches 0 while the numerator does NOT approach 0.

In this case, there is indeed an asymptote $x = -2$

Domain?

Range?

Which way does the graph "go" near an asymptote? Look at the sign!



Horizontal asymptotes

- If the numerator $p(x)$ has lower degree than the denominator $q(x)$, then the line $y = 0$ (the x-axis) is a horizontal asymptote
  Example: $G(x) = \frac{(x+2)}{x^2+4x+2}$ (previous example) – numerator degree is ___, denominator degree is ___
- If the numerator and denominator have equal degree, say n, then there is a horizontal asymptote $y = c$ where $C = \dfrac{coefficient\ of\ x^n\ in\ p(x)}{coefficient\ of\ x^n\ in\ q(x)}$
  Example: $G(x) = \frac{2x^2}{x^2+4x+2}$ then both p and q have degree 2.
  $c = \frac{2}{1} = 2$ so $y = 2$ is a horizontal asymptote

No horizontal asymptotes

- If the numerator has degree greater than the numerator, then the graph has no horizontal asymptote. To determine the "end-behavior" for large positive and negative x, find the ratio of the highest-degree terms in numerator and denominator

Example: $R(x) = \frac{-x^3+3x+5}{x+1}$

Ratio of highest-degree terms: $\dfrac{-x^3}{x} = -x^2$

The "end behavior" will be similar to $f(x) = -x^2$, a parabola opening down.