Review of Simple Interest

You borrow \$100 at an annual interest rate of 5%. How much would you owe at the end of 1 year

	Symbol/equation	**Example**
Principle	P	$\$100$
Interest rate	r	5% per year (annual)
Time	t	1 year
Interest amount	$I = Prt$	$I = \$100 * 5\% * 1 = \$5$
Amount owed	$A = P + I$ $A = P(1 + rt)$	$A = \$100 + \$5 = \$105$ $A = \$100(1 + 5\% * 1) = \$100 * (1.05) = \$105$

Suppose you borrow \$100 and owe \$108.50 at the end of 1 year? What is the interest rate?

If you owe \$120 at the end of two years and the interest rate is 6%,

- Which of P, A, I, r, t do you know?
- Which don't you know, and can you calculate it?

COMPOUND INTEREST

Interest rates for home or car loans are *annual* interest rates – but how often does one normally make payments on a car or a house?

Compounding interest every month:

- Divide the annual interest rate by 12 (this answers the question *what is the interest rate per month?*)
- Calculate the amount owed at the end of each month, and calculate interest for the NEXT month based on the total amount owed at the end of THIS month

Example: borrow $100,000 at 6% per year

100,000.00	Amount borrowed
6%	Annual interest rate
$\frac{6\%}{12} = 0.50\%$	Monthly interest rate

Month	Amount owed at start of month	Interest at end of month	Amount owed at end of month	Calculating amount owed
1	100,000.00	500.00	100,500.00	$P(1 + 0.5\%)$
2	100,500.00	502.50	101,002.50	$P(1 + 0.5\%)(1 + 0.5\%) = P(1 + 0.5\%)^2$
3	101,002.50	505.01	101,507.51	$P(1 + 0.5\%)^3$
4	101,507.51	507.54	102,015.05	
5	102,015.05	510.08	102,525.13	
6	102,525.13	512.63	103,037.75	
7	103,037.75	515.19	103,552.94	
8	103,552.94	517.76	104,070.70	
9	104,070.70	520.35	104,591.06	
10	104,591.06	522.96	105,114.01	
11	105,114.01	525.57	105,639.58	
12	105,639.58	528.20	106,167.78	$P(1 + 0.5\%)^{12}$

How much interest is *really* being paid?

100,000.00	Amount borrowed
6%	Annual interest rate
0.50%	Monthly interest rate
106,167.78	Amount owed at end of 1 year
6,167.78	Interest paid
6.1678%	Effective annual interest rate

If an amount P is borrowed for 1 year at annual interest rate of r, compounded n times per year, then the amount owed is

$A = P\left(1 + \frac{r}{n}\right)^n$

The *effective annual interest rate* is

$$\frac{\textit{interest paid during one year}}{\textit{amount owed at beginning of the year}} = \frac{A - P}{P} = \frac{A}{P} - 1 = \left(1 + \frac{r}{n}\right)^n - 1$$

# periods/year	effective rate
1	6.0000%
2	6.0900%
4	6.1364%
12	6.1678%
100	6.1817%
1000000	6.1837%

CONTINUOUSLY COMPOUNDED INTEREST

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$$

and

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r$$

$e^{0.06}$-1	6.1837%

Compound interest formula – multiple years:

	Symbol/equation	**Example**
Principle	P	$100
Interest rate	r	5% per year (annual)
Time	t	5 years
Number of periods per year	n	12
Amount owed	$A = P\left(1+\frac{r}{n}\right)^{nt}$	$A = \$100\left(1+\frac{0.05}{12}\right)^{12*5}$ $= \$100 * (1.283358679)$ $= \$128.34$

Continuously compounded for t years
$A = Pe^{rt}$

Terminology (same calculations!):

P	r	A	t	n
Principle	Annual interest rate	Amount	Term	Compounding periods per year Monthly: 12 Quarterly: Daily: Continuously?
Amount borrowed	Annual interest rate	Amount you owe	Loan term	
Amount loaned	Annual interest rate	Amount you will be paid back	Loan term	
Amount invested	Annual rate of return	Amount the investment will be worth	Duration of investment	
Present value	Annual interest rate	Future amount Future value	Duration of investment	

The *present value* of an amount you will have in the future is the principle that must be *invested* so that the amount paid back is equal to the future amount.

P. 470 #4

How much will you have in 3 years if you invest $50 at 6%, compounded monthly?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P

r

A

T

n

P. 470 #16

How much do you have to invest now to get $800 after 2.5 years, if the annual rate is 7% compounded monthly?

P

r

A

T

n

P. 470 #20

How much do you have to invest now to get $800 after 2.5 years, if the annual rate is 8% compounded continuously?

P. 470, #26

What rate of interest compounded annually is required to double an investment in 10 years?

P. 471 #34

If Angela has $100 to invest at 10% per annum compounded monthly, how long will it be before she has $175? If the compounding is continuous, how long will it be?

P. 471 #40

John will require $3000 in 6 months to pay off a loan that has no prepayment privileges. If he has the $3000 now, how much of it should he save in an account paying 3% compounded monthly so that in 6 months he will have exactly $3000?

P. 471 #42

42. Tracy is contemplating the purchase of 100 shares of a stock selling for $15 per share. The stock pays no dividends. Her broker says that the stock will be worth $20 per share in 2 years. What is the annual rate of return on this investment?

P. 471 #44

44. Tanya has just inherited a diamond ring appraised at \$5000. If diamonds have appreciated in value at an annual rate of 8%, what was the value of the ring 10 years ago when the ring was purchased?

46. On January 1, Kim places \$1000 in a certificate of deposit that pays 6.8% compounded continuously and matures in 3 months. Then Kim places the \$1000 and the interest in a passbook account that pays 5.25% compounded monthly. How much does Kim have in the passbook account on May 1?

48. Suppose that April has access to an investment that will pay 10% interest compounded continuously. Which is better: To be given $1000 now so that she can take advantage of this investment opportunity or to be given $1325 after 3 years?

EXPONENTIAL GROWTH

The population of deer in a certain area depends on the rate at which they reproduce (birth rate) and their death rate. Eg, for every 1000 deer present at the beginning of this year, 3 baby deer are born, and 10 deer pass away.

What is the *rate of growth?*

Beginning of year: 1000

End of year: 1000+20-10=1020

Typical practice in science is to express this in terms of the exponential function:

$N(0) = 1000$

$N(1) = 1010$

$$\frac{N(1)}{N(0)} = 1.01$$

$N(t) = N(0)e^{kt}$

How do we find k in the above?

$N(t) = N(0)e^{kt}$

$\frac{N(t)}{N(0)} = e^{kt}$

$\frac{1010}{1000} = e^{k*1}$

$1.01 = e^{k*1}$

$\ln(1.01) = k$ or $k = \ln(1.01) \approx 0.00995$

$N(t) = 1000e^{\ln(0.01)t} \approx 1000e^{0.00995t}$

How long will it take for the population to double?

EXPONENTIAL DECAY

After you take a dose of medicine, it's typical that a certain percentage of it is metabolized (turned into something else by your body) every hour.
Suppose you take 500 milligrams of Tylenol, and the body metabolizes 10% of it every hour. How long until there is only 250 milligrams of un-metabolized Tylenol?
Let $A(t)$ be the amount of Tylenol at time t.
$A(0) = 500$
$A(1) = 500(100 - 0.1) = 450$
$A(t + 1) = (100 - 0.1)A(t)$
$A(t) = A(0)e^{kt}$ but now k is *negative.*

Calculate k and also calculate the *half-life* of the medicine: the time at which only half the original amount is present:

NEWTON's Law of cooling

A hot (or cold) object is put into the atmosphere.
Let T be the temperature of the atmosphere (in the vicinity of the object)
$u(0) = u_0$be the initial temperature of the object
$u(t)$be the temperature of the object at time t

Newtons law is
$u(t) = T + (u_0 - T)e^{kt}$

P. 483

16. Thawing Time of a Steak A frozen steak has a temperature of 28°F. It is placed in a room with a constant temperature of 70°F. After 10 minutes, the temperature of the steak has risen to 35°F. What will the temperature of the steak be after 30 minutes? How long will it take the steak to thaw to a temperature of 45°F? [See the hint given for Problem 15.]